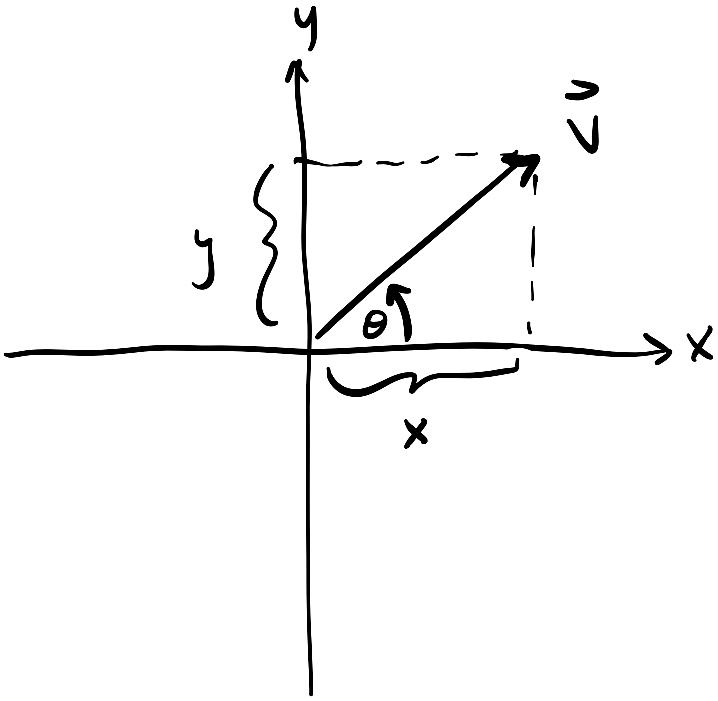
PHYS 231 - Oct. 16, 2023

Last Time:

2-D Vector



$\vec{v} = x\hat{i} + y\hat{j}$

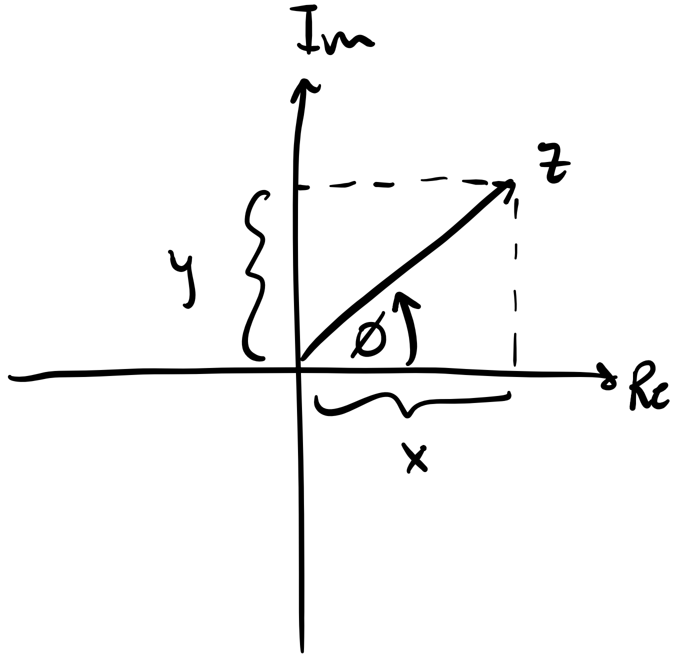
$|\vec{v}| = \sqrt{x^2 + y^2}$

$\tan\theta = \frac{y}{x}$

$x = |\vec{v}|\cos\theta$

$y = |\vec{v}|\sin\theta$

Complex Number



$z = x + jy$

$\text{Re}[z] = x$

$\text{Im}[I] = y$

$|z| = \sqrt{x^2 + y^2}$

$\tan\phi = \frac{y}{x}$

$x = |z|\cos\phi$

$y = |z|\sin\phi$

Taylor series: Claim, can express any fcn that is infinitely differentiable as a power series:

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

Need a scheme for finding the ~~ ~~ values of the coefficients $a_0, a_1, a_2, \dots, a_n, \dots$

Consider $f(0) = a_0 + \cancel{a_1 0}^{0} + \cancel{a_2 0^2}^{0} + \dots$

$$\boxed{\therefore a_0 = f(0)}$$

Next, consider df/dx

$$\frac{df}{dx} = 0 + a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots$$

then $\boxed{\left.\frac{df}{dx}\right|_{x=0} = a_1}$

Next, consider $\frac{d^2f}{dx^2}$ when $x=0$.

$$\frac{d^2f}{dx^2} = 0 + 2a_2 + 3.2a_3x + 4\cdot3\cdot a_4 x^2 + \ldots$$

$$\boxed{\frac{1}{2}\frac{d^2f}{dx^2}\bigg|_{x=0} = a_2}$$

One more... consider $\frac{d^3f}{dx^3}\bigg|_{x=0}$

$$\frac{d^3f}{dx^3} = 0 + 3\cdot2\cdot a_3 + 4\cdot3\cdot2\cdot a_4 x + \ldots$$

$$\boxed{\frac{1}{3\cdot2}\frac{d^3f}{dx^3}\bigg|_{x=0} = a_3}$$

In general, the n^{th} coefficient in a Taylor series expansion of $f(x)$ is given by:

$$\boxed{a_n = \frac{1}{n!} \left. \frac{d^{(n)} f}{dx^{(n)}} \right|_{x=0}}$$

Taylor series expansion of $f(x)$ is:

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \left. \frac{d^{(n)} f}{dx^{(n)}} \right|_{x=0} x^n}$$

Example: Find Taylor series of $\sin x$.

n	a_n	[redacted] $\frac{d^{(n)}}{dx^{(n)}} \sin(x)$	$\left. \frac{1}{n!} \frac{d^{(n)}}{dx^{(n)}} \sin x \right\|_{x=0}$
0	a_0	$\sin x$	0
1	a_1	$\cos x$	1
2	a_2	$-\sin x$	0
3	a_3	$-\cos x$	$-1/3!$

4	a_4	$\sin x$	0
5	a_5	$\cos x$	$1/5!$
6	a_6	$-\sin x$	0
7	a_7	$-\cos x$	$-1/7!$

$$\vdots$$

$$\sin x \simeq x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \cdots$$

Exercise for the student:

Show that the Taylor series expansions for $\cos x$ & e^x are:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

To connect Taylor series to complex nos, consider the Taylor series expansion of $e^{j\phi}$. That's set $x = j\phi$ in #

$$e^{j\phi} = 1 + j\phi + \frac{(j\phi)^2}{2!} + \frac{(j\phi)^3}{3!} + \frac{(j\phi)^4}{4!} + \frac{(j\phi)^5}{5!} + \cdots$$

n	j^n	
0	j^0	1
1	j^1	j
2	j^2	-1
3	j^3	$-j$
4	j^4	1
5	j^5	j

$$\therefore e^{j\phi} = 1 + j\phi - \frac{1}{2!}\phi^2 - j\frac{1}{3!}\phi^3 + \frac{1}{4!}\phi^4 + j\frac{1}{5!}\phi^5 - \cdots$$

$$e^{j\phi} = \overbrace{\left(1 - \frac{1}{2!}\phi^2 + \frac{1}{4!}\phi^4 - \cdots\right)}^{\cos\phi} + j\underbrace{\left(\phi - \frac{1}{3!}\phi^3 + \frac{1}{5!}\phi^5 - \cdots\right)}_{\sin\phi}$$

Euler's Eq'n: $e^{j\phi} = \cos\phi + j\sin\phi$

(Exercise for the student: Show that $e^{-j\phi} = \cos\phi - j\sin\phi$)

Recall that we can express a complex no. as

$$z = x + jy$$

$|z| = \sqrt{x^2+y^2}$

$\tan\phi = y/x$

$x = |z|\cos\phi$

$y = |z|\sin\phi$

$$\therefore z = |z|\cos\phi + j|z|\sin\phi$$

$$= |z|\underbrace{(\cos\phi + j\sin\phi)}_{e^{j\phi}}$$

Euler's Eq'n

$$\boxed{z = x + jy \Longleftrightarrow z = |z|e^{j\phi}}$$

Fun w/ Euler's Eq'n

$$e^{j\phi} = \cos\phi + j\sin\phi$$

Try $\phi = \pi$

$$e^{j\pi} = \underbrace{\cos\pi}_{-1} + j\underbrace{\sin\pi}_{0}$$

$$\boxed{-1 = e^{j\pi}}$$

Try $\phi = 2\pi$

$$e^{j2\pi} = \underbrace{\cos 2\pi}_{+1} + j\underbrace{\sin 2\pi}_{0}$$

$$\boxed{1 = e^{j2\pi}}$$

Try $\phi = \frac{\pi}{2}$

$$e^{j\pi/2} = \underbrace{\cos\frac{\pi}{2}}_{0} + j\underbrace{\sin\frac{\pi}{2}}_{+1}$$

$$j = e^{j\pi/2}$$

We can now try to evaluate $\sqrt{j}$

$$\sqrt{j} = \sqrt{e^{j\pi/2}} = \left(e^{j\pi/2}\right)^{1/2}$$

$$\therefore \sqrt{j} = e^{j\pi/4}$$

Now use Euler's eq'n again to write:

$$\sqrt{j} = \cos\frac{\pi}{4} + j\sin\frac{\pi}{4}$$

$$= \frac{1}{\sqrt{2}}(1+j)$$

Test

$$\underbrace{\left[\frac{1}{\sqrt{2}}(1+j)\right]}_{\sqrt{j}}^{2} = \frac{1}{2}(1+j)(1+j)$$

$$= \cancel{\tfrac{1}{2}}\left(\cancel{1} + \cancel{2}j + \cancel{\underbrace{j^2}_{-1}}\right)$$

$$= j \checkmark$$

Last Definition: Complex Conjugate.

The complex conjugate of complex num.
$z = x + jy = |z|e^{j\phi}$ is defined to
be:

$$z^* = x - jy = |z|e^{-j\phi}$$

"complex conjugate of z"

just reverse sign of every j factor.